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The fluid flow due to two submerged sinks in a two layer stratified fluid

X. Wen
School of the Environment, University of Leeds, Leeds, UK
K. Manik
Centre for Computational Fluid Dynamics, University of Leeds, Leeds, UK

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Abstract *A boundary integral technique is developed to study the free surface flow of a steady, two-dimensional, incompressible, irrotational and inviscid fluid flow which is produced by two submerged sinks (or sources) which are of equal strength, placed along a solid horizontal boundary with a stagnation point on the free surface in a two layer stratified fluid in the presence of gravity. A special form of the Riemann-Hilbert problem, namely the Dirichlet boundary problem, is applied in the derivation of the governing non-linear boundary integral-differential equations which are solved for the fluid velocity on the free surface and this involves the use of an interpolative technique and an iterative process. Results have been obtained for the free surface flow for various values of the Froude number and sink locations on the solid horizontal boundary and we have also studied the largest value of the Froude number for which no convergent solutions are possible, namely the critical Froude number. We have found that the free surface profile is dependent on two parameters, namely the Froude number on the free surface and the non-dimensional distance between the two sinks.*

Introduction

The study of free surface fluid flows which are induced by submerged sinks or sources have been the subject of considerable research by both fluid dynamicists and engineers due to the numerous engineering applications of stratified flows. Examples of common stratified flows are in the cooling of power stations where water is stored in cooling ponds which consist of a warm upper layer and a cooler and more dense lower layer which is pumped into the power station cooling system; stratified flows in the field of crude oil extraction from underground reservoirs; etc. In such problems, it is important for the engineer to be able to know the geometry of the free surface in advance so that the fluid may be withdrawn with maximum efficiency and stability. For further examples of selective withdrawal, see Imberger (1980), Imberger and Hamblin (1982) and Yih (1980).

The fluid flow induced by a submerged sink in a two layer fluid, where the sink is placed below the free surface, is a classical free surface flow problem. A two layered fluid comprises of two distinct fluids with different densities

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where the less dense fluid is on the top of the more dense fluid. There has been a wide variety of research papers published on this topic and both numerical and experimental investigations performed, see Hocking and Forbes (1992) and Vanden-Broeck *et al.* (1995) for further information on the two layer problem and Wen and Ingham (1992) for the three layer problem.

In this paper, a boundary integral technique is developed which is used to study the free surface flow of a steady, two-dimensional, incompressible, irrotational and inviscid fluid which arises due to two submerged sources (or sinks) which are situated along a solid horizontal boundary in the presence of gravity. Below the solid horizontal boundary, the fluid consists of two homogeneous layers, each of different density and separated by a free surface interface. There appears to be no previous research performed in which two, or more, submerged sinks or sources are placed along a solid horizontal boundary in a two layer fluid flow problem when there is a stagnation point on the free surface. However, Debler and Meyerinck (1980) performed an experiment using a holographic interferometer technique for the three-dimensional fluid flow into two sinks in a two layer fluid and in this case the fluid flow was supercritical, i.e. $\mathrm{Fr} > 1$, and was characterised by the formation of cusp points on the free surface in the vicinity just above the sinks.

The present numerical procedure for the solution of this type of problem is different to most of the previous researchers in this field since it involves the application of a special case of the Riemann-Hilbert problem, namely the Dirichlet boundary problem (Muskhelishvilli, 1953) in the derivation of the non-linear boundary integral-differential equations. Further, this boundary integral technique is inherently different from the traditional boundary element method in that it does not require the inversion of a large system of non-linear algebraic equations and therefore requires less computational memory and CPU time. The non-linear boundary integral equations are then solved using a piecewise interpolation technique and an iterative process. The aim of this paper is to study the effect of the free surface flow induced by two submerged sinks (or sources) placed along a solid horizontal boundary with a stagnation point on the free surface in the presence of gravity.

Formulation of the boundary integral equation

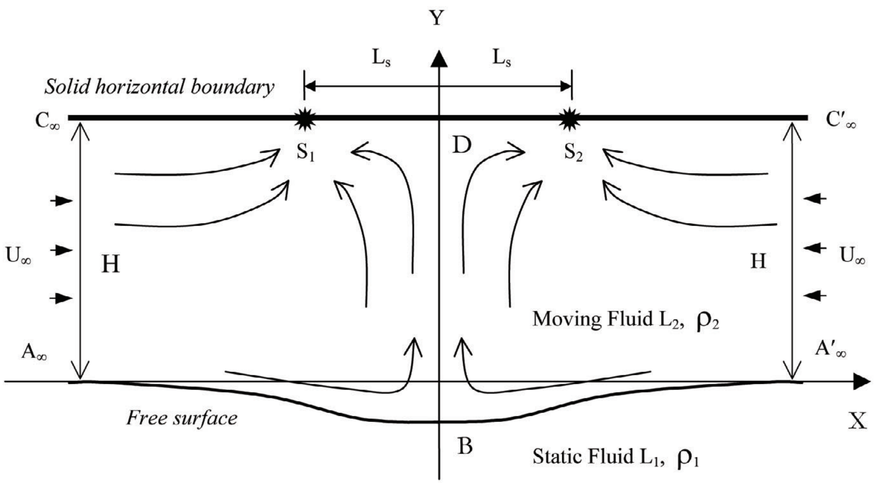
Consider the case of a steady, incompressible, inviscid, two-dimensional fluid which comprises of two distinct layers L_1 and L_2, each of different constant densities, namely ρ_1 and ρ_2. The layer L_2 corresponds to the uppermost layer which is the less dense fluid, say warm water or oil, L_1 is the lowermost layer, say cold water. Hence, we have $\rho_2 < \rho_1$ and the only force considered to be acting on the fluid is gravity. The lowermost layer L_1 is of infinite depth and occupies a semi-infinite region, namely the region below $A_\infty A'_\infty$, see Figure 1 which represents the physical plane of the two layer fluid flow induced by two submerged sinks. The upper layer L_2 is of finite depth and occupies the infinite strip $A_\infty BA'_\infty C_\infty DC_\infty$. Above the two layer fluid system, two sinks of equal strengths $2Q$ are placed along a solid horizontal boundary $C_\infty C'_\infty$ (say a layer of ice) at the points S_1 and S_2, respectively, equal distance from the point D. Since the sinks are of equal strength and are at equal distances from the Y-axis, i.e. at $(\pm L_s,\ Y_D)$, the flow can be assumed to be symmetrical about the Y-axis and hence we only need to consider the solution of the free surface in the left-hand-side of the X-Y plane, i.e. $X \leq 0$. The fluid flow on the free surface is characterised by a stagnation point at the point B where the free surface falls to its minimum elevation. There is no fluid flow in the lower layer and hence the fluid is stagnant in this layer. The geometry of the free surface is not known *a priori* but the Bernoulli equation provides a non-linear boundary condition on the free surface flow in the upper layer which can be used to calculate the free surface elevation, once the free surface velocity is known.

The complex potential is denoted by $W = \Phi + \mathrm{i}\Psi$, where Φ is the velocity potential and Ψ is the stream function and it is analytic in the fluid domain $A_\infty BDC_\infty$.

We now introduce the non-dimensionalisation:

$$z = x + \mathrm{i}y = (X + \mathrm{i}Y)/H, \quad w = \varphi + \mathrm{i}\psi = (\Phi + \mathrm{i}\Psi)/U_\infty H,$$
$$q = Q/(U_\infty H) \ \text{and} \ \ell_s = L_s/H \tag{1}$$

and let the non-dimensional stream function ψ be $\psi = 0$ on the lower boundary and $\psi = 1$ on the upper boundary. In the physical plane, the two layer stagnation point fluid flow induced by a two submerged sinks in terms of the complex



Note: The position of the sinks are at (X, Y) = ($-L_s$, Y_D) and (L_s, Y_D) and there is a stagnation point on the free surface at the point B, i.e. at the point (0, Y_B)

Figure 1. The physical plane for the flow induced by two sinks of equal strengths situated along a solid horizontal boundary $C_\infty C'_\infty$ in a two layer fluid flow configuration, where L_1, and L_2 are two distinct layers, each of different density ρ_1 and ρ_2, where $\rho_2 < \rho_1$

z-plane, with the x-axis being horizontal and the y-axis being vertical, and the points A_∞ and C_∞ located at $x = -\infty$ and the points B and D located at $x = 0$. The boundaries $A_\infty B$ and $C_\infty D$ are the free surface and the solid boundary, respectively. By the application of a Schwartz-Christoffel mapping function, the infinite strip $A_\infty BS_1DC_\infty$ in the complex potential w-plane is transformed onto the upper half-plane of the auxiliary t-plane. The angle that the free surface $A_\infty B$ makes with the horizontal is given by $\theta = \theta^-$, which maps onto the negative real axis of the t-plane as indicated by the superscript $-$. The angle that the solid boundary $C_\infty S_1$ makes with the horizontal is given by $\theta = \theta^+$, where the superscript + indicates that the boundary $C_\infty S_1$ maps onto the positive real axis of the t-plane. The tangential component of the non-dimensional fluid velocity on the free surface and solid boundary are given by u^- and u^+, respectively. From this point onwards, we let any variable with a superscript $(-)$ indicate that the variable is related to the free surface $A_\infty B$ (or the negative real t-plane) and the superscript (+) indicate that the variable is related to the solid boundary $C_\infty S_1$ (or the positive real t-plane).

The fluid velocity at any point on the free surface satisfies the Bernoulli equation and can be written in non-dimensional form,

$$\frac{\mathrm{Fr}^2}{2}(u^-)^2 - y^-(x) = \frac{\mathrm{Fr}^2}{2} \quad \text{on } A_\infty B \tag{2}$$

where Fr is the Froude number on the free surface and is defined by

$$\mathrm{Fr} = U_\infty/\sqrt{gH} \tag{3}$$

The region occupied by the fluid flow, namely $A_\infty BDC_\infty$, in the z-plane, is mapped onto the w-plane and, without any loss in generality, the free surface $A_\infty B$ and the solid boundary $C_\infty D$ in the physical plane are mapped in the w-plane onto the streamlines $\psi = 0$ and 1, respectively. The logarithm of the transformation of the complex velocity Ω, given by the equation $\mathrm{d}w/\mathrm{d}z = u\mathrm{e}^{-\mathrm{i}\theta}$, is given by

$$\Omega = \ell\mathrm{n}(\mathrm{d}w/\mathrm{d}z) = \tau - \mathrm{i}\theta \tag{4}$$

where u is the non-dimensional fluid speed at some point in the flow field, θ is the angle that the fluid velocity vector makes with the positive x-axis and $\tau = \ell\mathrm{n}(u)$, $\mathrm{d}w/\mathrm{d}z$ and Ω are analytic functions within the infinite strip $A_\infty BS_1DC_\infty$ shown in the w-plane and whose imaginary part, $\mathscr{I}m\{\Omega(t)\} = -\theta$, is related to the angle that the solid boundary and the free surface makes with the horizontal and its real part, $\mathscr{R}e\{\Omega(t)\} = \tau$, is related to the fluid velocity vector on the solid boundary and the free surface (Figure 2(a)).

The infinite strip $A_\infty BS_1DC_\infty$ in the complex potential w-plane is transformed onto the upper half-plane of the auxiliary t-plane, $t = \eta + \mathrm{i}\xi$, by applying the Schwartz-Christoffel mapping function given by

$$t = -\mathrm{e}^{-\pi w} \tag{5}$$

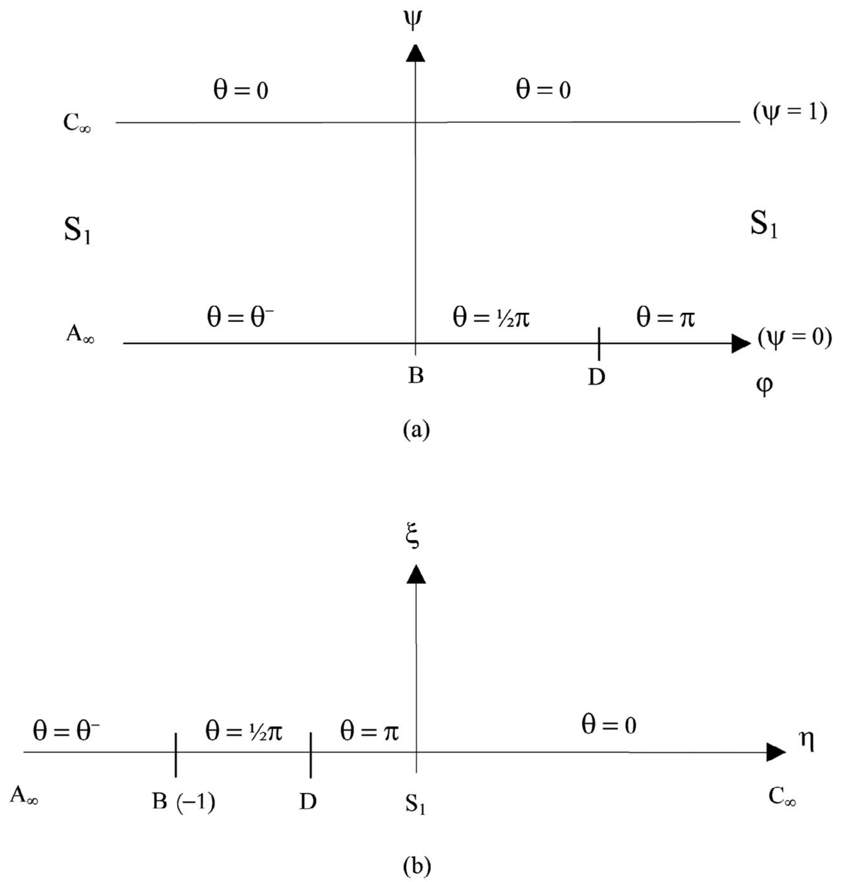


Figure 2.
(a) The w-plane for the fluid flow due to a submerged sink S_1 along a solid horizontal boundary in a two layer fluid, and (b) the t-plane for the fluid flow due to a submerged sink S_1 placed along a solid horizontal boundary in a two layer fluid

The transformed plane is shown in Figure 2(b) and the Schwartz-Christoffel mapping function, namely equation (5), transforms the domain occupied by the fluid in the infinite strip $0 \leq \psi \leq 1$ in the w-plane, to the upper half of the t-plane. The solid boundary $C_\infty D$ corresponds to the positive real t-plane, $\eta > \eta_d$ and the free surface $A_\infty B$ of the flow domain corresponds to remainder of the t-axis, $\eta < \eta_b$.

The dynamic boundary condition on the free surface $A_\infty B$ is given by the Bernoulli equation (2) on $A_\infty B$. On the fixed solid boundary $C_\infty D$, the fluid velocity component normal to the surface is zero. The required variables, namely the fluid velocity on the free surface boundary, can be used to solve a Dirichlet boundary-value problem. The Dirichlet method for a boundary-value problem in the upper plane is introduced to express Ω as a function of t. For the Dirichlet boundary-value problem, the boundary conditions on the real η-axis of the t-plane are as follows:

$$\text{On } A_\infty B \ (\eta < \eta_b), \quad \mathscr{I}m\,\Omega(\eta) = -\theta^-(\eta) \tag{6}$$

$$\text{On } BD \ (-1 < \eta < \eta_d), \quad \mathscr{I}m\ \Omega(\eta) = -\pi/2 \tag{7}$$

$$\text{On } DS_1 \ (\eta_d < \eta < 0), \quad \mathscr{I}m\ \Omega(\eta) = -\pi \tag{8}$$

$$\text{On } C_\infty D \ (\eta > 0), \quad \mathscr{I}m\ \Omega(\eta) = 0 \tag{9}$$

We express the logarithmic variable Ω, given by equation (4), as a function of the single variable t and the problem then reduces to a Dirichlet boundary-value problem in the upper half-plane. The Dirichlet boundary-value problem states that when the imaginary part of an analytical function $\Omega(t) = \tau(t) - \mathrm{i}\theta(t)$ on the real axis η of the upper half plane is known, i.e. the angle that the free surface makes with the horizontal is given, i.e. $\mathscr{I}m\ \Omega(\eta) = -\theta(\eta)$, and it satisfies the Holder condition at $\eta = \eta_o$ for all values of η on the real axis sufficiently close to η_o, namely

$$|\theta(\eta) - \theta(\eta_o)| \leq A|\eta - \eta_o|^\mu \ \text{ and } \ 0 < A < \infty, \ \ 0 < \mu \leq 1 \tag{10}$$

then the solution is found by referring to the work of Muskhelishvilli (1953) and is given by the Schwarz formula, namely

$$\Omega(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\eta)}{\eta - t}\,\mathrm{d}\eta + \Omega_\infty \tag{11}$$

where Ω_∞ is the value of the function $\Omega(t)$ at $t = \infty$ and since $\tau(\infty) = 0$ and $\theta(\infty) = 0$, then this implies that $\Omega_\infty \equiv 0$. Since we know the angle that the free surface makes with the horizontal, we only have to find the fluid velocity on the free surfaces. On letting t approach the point η_o on the real axis from the upper half plane and taking the Cauchy principal value, we obtain

$$\Omega(\eta_o) = \tau(\eta_o) - \mathrm{i}\theta(\eta_o) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\eta)}{\eta - t}\,\mathrm{d}\eta - \mathrm{i}\theta(\eta_o) \tag{12}$$

Since we only require the fluid velocity on the free surface, we compare the real parts of equations (4) and (12), and by using the boundary conditions (6)-(9), and after some algebraic manipulations of equation (12), we find that the fluid velocity on the free surface is given by

$$\ell\mathrm{n}(u(\eta_o)) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\theta^-(\eta)}{\eta - \eta_o}\,\mathrm{d}\eta - \frac{1}{2}\,\ell\mathrm{n}\left|\frac{(\eta_o - \eta_d)}{(1 + \eta_o)}\right| - \ell\mathrm{n}\left|\frac{\eta_o}{(\eta_o - \eta_d)}\right| \tag{13}$$

which gives the fluid velocity, $u(\eta_o)$, on the free surface $C_\infty D$, which corresponds to the region given by $\eta > \eta_d$ in the t-plane. We see from equation

(13) that there exist three singularities, namely, one at $\eta_o = \eta_d$, which corresponds to the stagnation point D, another at $\eta_o = -1$, which corresponds to the stagnation point B and finally at $\eta_o = 0$, which corresponds to the submerged sink at the point S_1 in the physical plane.

Now, we introduce the independent variable s, which is taken to be the arc length along the boundaries, and apply equation (5) to the obtained relations that connect the t-plane to the physical plane for the free surface boundaries, namely

$$\begin{aligned} \mathrm{d}\eta &= \pi \mathrm{e}^{-\pi\varphi(s)} u(s)\,\mathrm{d}s \ \text{ on } \Gamma_1 \\ \mathrm{d}\eta &= -\pi \mathrm{e}^{-\pi\varphi(s)} u(s)\,\mathrm{d}s \ \text{ on } \Gamma_2 \end{aligned} \qquad \text{(14(a) and (b))}$$

where the free surface and the solid boundary are denoted by Γ_1 and Γ_2, respectively. Along the free surface we have

$$u(s) = \mathrm{d}\varphi(s)/\mathrm{d}s \qquad (15)$$

which on integration gives

$$\varphi(s) = \varphi_B + \int_0^s u(\ell)\,\mathrm{d}\ell \ \text{ on } \Gamma_1 \qquad (16)$$

$$\varphi(s) = \varphi_D + \int_0^s u(\ell)\,\mathrm{d}\ell \ \text{ on } \Gamma_2 \qquad (17)$$

where we take the value of the potential functions at the points B and D to be φ_B and φ_D, respectively. In the physical plane we may write equation (13) to give the fluid velocity on the free surface as

$$\ell\mathrm{n}(u(s)) = -\int_{\Gamma_1} \frac{\theta^-(\ell)\eta(\ell)u(\ell)}{\eta(\ell)-\eta_o(s)}\,\mathrm{d}\ell - \frac{1}{2}\ell\mathrm{n}\left|\frac{(\eta_o(s)-\eta_d)}{(1+\eta_o(s))}\right| - \ell\mathrm{n}\left|\frac{\eta_o(s)}{(\eta_o(s)-\eta_d)}\right| \qquad (18)$$

where $\eta(s)$ and $\eta_o(s)$ are given implicitly by

$$\eta(\ell) = -\mathrm{e}^{-\pi\varphi(\ell)}, \quad \eta_o(s) = -\mathrm{e}^{-\pi\varphi(s)} \ \text{ on } \Gamma_1 \qquad (19)$$

$$\eta(\ell) = -\mathrm{e}^{-\pi\varphi(\ell)}, \quad \eta_o(s) = -\mathrm{e}^{-\pi\varphi(s)} \ \text{ on } \Gamma_2 \qquad (20)$$

and $\varphi(\ell)$ and $\varphi(s)$ are the velocity potentials at any general point and the specific point $\ell = s$, respectively. The solution of the boundary integral equation can be obtained on the boundary Γ_1 in the physical plane, where we know the function $\theta^-(s)$. The unknown variables are the fluid velocity $u(s)$ and the (x, y) co-ordinates of the free surface and these variables are found by an iterative procedure as described in the next section.

The position of the sink S_1 from the point D is calculated by using the numerical technique which involves the application of equation (13) for the fluid velocity, u, on the free surface and the potential function φ given by equation (15). The value of the velocity potential φ_D at the point D determines the value of ℓ_s and the value of φ_D is fixed and it is chosen appropriately to give the desired value of ℓ_s. The position of the sink is found by the use of equation (15) which relates the arc length of the free surface, s, in the physical plane to variables in the upper half of the t-plane, i.e.

$$\mathrm{d}s_\mathrm{o} = \frac{\mathrm{d}\eta_\mathrm{o}}{\pi u(\eta_\mathrm{o})\eta_\mathrm{o}} \tag{21}$$

Substitution of equation (13) for $u(\eta_\mathrm{o})$ in equation (21) yields the position of the sink, ℓ_s, from the point D as follows:

$$\ell_s = \frac{1}{\pi}\int_{\eta_\mathrm{d}}^{0} \frac{\mathrm{e}^{\left\{\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\theta^-(\eta)}{\eta-\eta_\mathrm{o}}\mathrm{d}\eta\right\}}}{\sqrt{(1+\eta_\mathrm{o})(\eta_\mathrm{o}-\eta_\mathrm{d})}}\,\mathrm{d}\eta_\mathrm{o} \tag{22}$$

where the integral in the numerator can be estimated by approximating $\theta^-(\eta) = (\theta_i^- + \theta_{i+1}^-)/2$. It should be noted from equation (22) that the sink distance ℓ_s from the point D is dependent on the value of η_d or φ_D and it is also dependent on the angle that the free surface makes with the horizontal, θ^-. Therefore any calculation of the sink distance from the point D is performed once a convergent solution is obtained.

Numerical and iterative method for the boundary integral equation

In order to evaluate the integrals that occur in equation (18), we use a similar interpolative technique to that described by Wen and Manik (2000). We discretise the integrals in equation (18) over N small intervals $[\eta_i,\ \eta_{i+1}]$ and approximate the unknown function using a piecewise constant interpolation technique which enables us to perform analytical integrations over a small interval $[\eta_i,\ \eta_{i+1}]$ as follows. We use a piecewise constant procedure and over each sub-interval $[\eta_i,\ \eta_{i+1}]$ where there is no singularity point in the line of integration, i.e. $\eta_k \notin [\eta_i,\ \eta_{i+1}]$ where $\eta = \eta_k$ is the singularity point, we replace the unknown function $\theta^-(\eta)$ by constant interpolation, i.e. $\theta^-(\eta) = (\theta_i^- + \theta_{i+1}^-)/2$. If the singularity occurs at $\eta = \eta_k$ in the interval $[\eta_i,\ \eta_{i+1}]$ then we approximate $\theta^-(\eta)$ by $\theta^-(\eta) = \theta_k^-$, where θ_k^- is the angle that the free surface makes with the horizontal at the singularity point $\eta = \eta_k$.

The iterative numerical procedure employed to solve equations (15) and (18) is as follows.

(1) For the first iteration, we assume the geometry of the free surface to be initially in the form of a horizontal straight line, namely $y^{-}(s) = 0$. Hence, the angle that the free surface makes with the horizontal is known, i.e. $\theta(s) = 0$. We then discretise the x-axis of the boundaries $A_\infty B$ and $C_\infty D$ into N grid points, where the angle $\theta(s)$ that the free surface makes with the horizontal at each grid point is known.

(2) For the first iteration, we assume the initial fluid velocity $u^{n}(s)$ on the free surface $A_\infty B$ to be given by $u^{0}(s) = 1$ where the superscript n denotes the number of iterations.

(3) The velocity potential $\varphi(s)$ on the free surface $A_\infty B$ and the solid boundary $C_\infty D$ can be found by numerically integrating equation (15), i.e.

$$\varphi^{(0)} = \varphi^{(0)}(s_B) + \int_{s_B}^{s} u^{(0)}(\ell)\,\mathrm{d}\ell \quad \text{on } \Gamma_1 \tag{23}$$

$$\varphi^{(0)} = \varphi^{(0)}(s_D) + \int_{s_D}^{s} u^{(0)}(\ell)\,\mathrm{d}\ell \quad \text{on } \Gamma_2 \tag{24}$$

(4) We find the values of $\eta(s)$ on the t-plane using equations (19) and (20).

(5) The new fluid velocity distribution $u^{n+1}(s)$ on the free surface $A_\infty B$ is found by substituting the values of $\theta^{0}(s)$ and $\eta^{0}(s)$ into the right hand side of equation (18).

(6) The new fluid velocity $u^{n+1}(s)$ on the free surface is used to calculate the $y^{-}(s)$ co-ordinate of the free surface using equation (2).

(7) The new angle $\theta^{n+1}(s)$ that the free surface makes with the horizontal is calculated using a central-difference method, namely

$$\theta_i(s) = \tan^{-1}\left(\frac{y_{i+1}(s) - y_{i-1}(s)}{x_{i+1}(s) - x_{i-1}(s)}\right) \tag{25}$$

(8) The new fluid velocity distribution $u^{n+1}(s)$ is used as the new guess for the fluid velocity on the free surface $A_\infty B$ and inserted in step (2).

(9) The iterative process is repeated until the fluid velocity $u(s)$ on free surface $A_\infty B$ and the velocity potential $\varphi(s)$ have converged to within the required level of accuracy, say ε, such that

$$|u^{n+1}(s) - u^{n}(s)| < \varepsilon, \ \ |\varphi^{n+1}(s) - \varphi^{n}(s)| < \varepsilon \quad \text{on } \Gamma_1 \tag{26}$$

Numerical results and discussion

Calculations have been performed for the free surface profiles in the presence of two sinks for various values of the Froude number and sink distances ℓ_s from the point D in a two layer fluid. Initially, we study the effect on the free surface of increasing the Froude number with the sink distance $\ell_s = 1$ from the point D and then examine the free surface for various values of ℓ_s for given Froude numbers.

For most of the cases we have considered in this paper, we have found that a mesh size of $\Delta s = 0.015$ and the number of points on the free surface of $N = 400$ was sufficient to give graphically indistinguishable results when larger values of N and smaller values of Δs were employed. However, it is observed that the values of Δs and N necessary to give such results were dependent on the values of the upstream Froude numbers. Further, we have studied the critical Froude numbers, i.e. the largest value of the Froude number for which convergent solutions are possible, and it was found necessary to use a larger value of N and a smaller value of Δs in order to accurately determine and compute the free surface profiles. The number of grid points N was also dependent on the value of the distance ℓ_s of the sink from the point D. To find solutions for large values of ℓ_s, the range of values of x, i.e. the length of the line $C_\infty D$ also increases in order to obtain the full geometry of the free surface profile, therefore requiring a larger number of grid points N, e.g. for $\ell_s \approx 6$ and Fr ≈ 0.1 we require $N = 800$ and $\Delta s = 0.015$ when using piecewise constant interpolation. We found that the iterative procedure was convergent and, in general, it required less than about 20 iterations to obtain results with an accuracy of $\varepsilon = 10^{-5}$. This level of accuracy was found to be sufficient since any smaller value of ε produced results which were graphically indistinguishable.

Fluid flow for various Froude numbers

Figure 3 shows the free surface flow induced by two sinks of equal strengths, i.e. $q_1 = q_2 = 1$, non-dimensionalised with respect to $U_\infty H$, in a two layer fluid

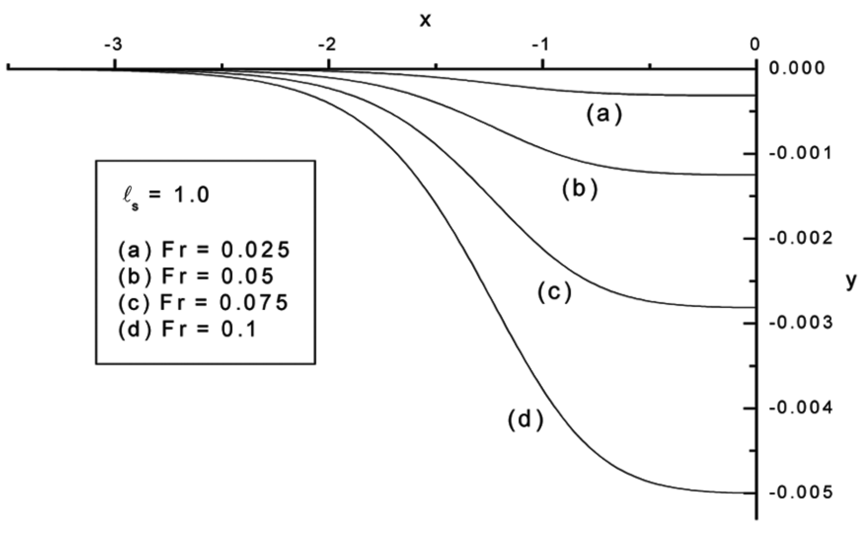


Figure 3.
The free surface flow induced by two sinks of equal strength in a two layer fluid situated on a solid horizontal boundary at $(x, y) = (-1, 1)$ and $(1, 1)$, where $\ell_s = 1.0$ for upstream Froude numbers Fr = 0.025, 0.05, 0.075 and 0.1

and situated on a solid boundary at $y = 1.0$. The sinks are situated at the points $(x, y) = (-\ell_s, 1)$ and $(\ell_s, 1)$, respectively, for Froude numbers $\mathrm{Fr} = 0.025$, 0.05, 0.075 and 0.1, and there is a stagnation point on the free surface. We observe, since the sink strengths are equal, that the free surface is symmetrical about the line $x = 0$ and therefore we only present the free surface in the region $x \leq 0$. In all cases, far upstream the free surface depth is unity and the surface falls gently until the flow approaches the sinks S_1 and S_2. In general, we observe that for all values of Fr considered, the free surface depths increase monotonically until the free surface reaches a maximum depth. The free surface depth increases most rapidly in the region close to the vicinity of the x location of the sinks, i.e. $-2.0 < x < -1.0$. It is interesting to note that the free surface reaches a maximum depth directly below the point $x = 0$, at which point the fluid velocity is zero. At this stagnation point we can calculate the exact position of the free surface using the Bernoulli equation (2). Further, we observe that an increase in the Froude number from $\mathrm{Fr} = 0.05$ to 0.1 has the effect of increasing the curvature of the free surface and if we set the Froude number on the free surface to be zero, i.e. $\mathrm{Fr} = 0$, then from the Bernoulli equation (2), the free surface is the horizontal line, $y^-(x) = 0$. We observe that when the upstream Froude number is $\mathrm{Fr} = 0.025$, then the free surface is close to being horizontal and is relatively unaffected by the presence of the sink. As the upstream Froude number increases from 0.025 to 0.1, we observe that both the free surface curvature and depth increase and this becomes more pronounced at larger values of Fr, especially in the region $-2.0 < x < -1.0$. We find that the critical value of the Froude number above which no convergent solution can be found is approximately 0.252 and it should be noted that near this critical Froude number there are no signs of oscillations on the free surface which may suggest the presence of waves.

Figure 4 shows the free surface fluid velocity u for the free surfaces shown in Figure 3, induced by a sink on a solid horizontal boundary at $y = 1.0$ and $\ell_s = 1.0$, for upstream Froude numbers $\mathrm{Fr} = 0.025$, 0.05, 0.075 and 0.1 in a two layer fluid. For all values of Fr investigated, we observe that the free surface velocity profiles are graphically indistinguishable. The reason being that the change in the free surface depth, as shown in Figure 3, is relatively small, being in the range 0.0-0.005, and therefore the effect on the free surface velocity is also small. Further, as we would expect, for all the values of Fr investigated, the free surface velocity far upstream is unity and decreases to zero at the stagnation point $x = 0$, with the most rapid change in the free surface velocity occurring in the region close to the x location of the sink, i.e. $-2.0 < x < -1.0$.

Fluid flow for various sink distances ℓ_s

Figure 5 shows the free surface flow induced by two sinks of equal strength on a solid boundary at $y = 1.0$ when the sink distances are $\ell_s = 0$, 1.0, 2.0 and 3.0 for $\mathrm{Fr} = 0.05$ and 0.1, in a two layer fluid. We observe that the general behaviour of the free surface profiles in Figure 5(a) and (b) are very similar, except that the depths of the free surfaces in Figure 5(b) are greater since the Froude number used in Figure 5(b) is larger, i.e. $\mathrm{Fr} = 0.1$. We observe from Figure 5(a) that increasing ℓ_s has a very important effect on the free surface profiles. For the case when $\ell_s = 0$, we find that the free surface is relatively close to the y-axis, as opposed to the situation when $\ell_s = 1.0$, 2.0 and 3.0. The greatest variation of the curvature in the free surface for $\ell_s = 0$ occurs in the region $-1.5 < x < -0.5$. As we increase ℓ_s, i.e. move the sink more in the negative direction, we find that the free surface profile is also "moved" or translated in the negative direction by a factor approximately proportional to ℓ_s. It is also interesting to note, the case $\ell_s = 2.0$, where the free surface becomes almost horizontal at $x \approx -1.0$, and for $\ell_s = 3.0$, the free surface becomes horizontal at $x \approx -2.0$. This is in contrast to the case when $\ell_s = 0$, where the free surface depth continues to increase until it reaches the stagnation point. We find that the values of the critical Froude number are approximately 0.290, 0.252, 0.201 and 0.178 when the sink distances are $\ell_s = 0$, 1.0, 2.0 and 3.0, respectively, and at the critical Froude number there are no signs of waves on the free surface. It is interesting to note that the maximum value of the critical Froude number occurs when $\ell_s = 0$ and we observe that on increasing ℓ_s, i.e. as the sink moves further away from the point D, this has the effect of reducing the critical Froude number. This phenomena could be due to the fact that as the sink moves away from the point D, the region in which the free surface velocity is close to zero, i.e. the fluid is stagnant, also increases, (Figure 6). This may create an instability in the two layer system at which point

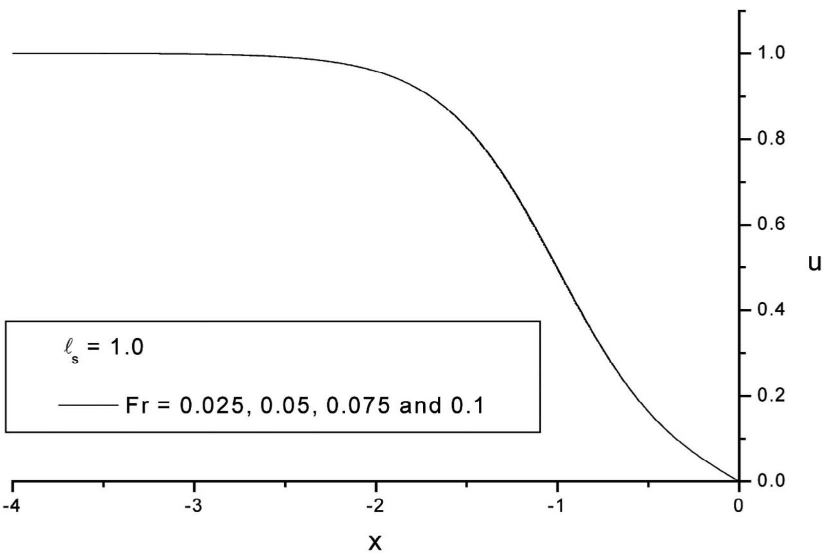


Figure 4.
The free surface velocity u for the free surfaces shown in Figure 3 induced by two sinks of equal strength in a two layer fluid situated on a solid horizontal boundary at $y = 1.0$ and $\ell_s = 1.0$ for upstream Froude numbers $\mathrm{Fr} = 0.025$, 0.05, 0.075 and 0.1

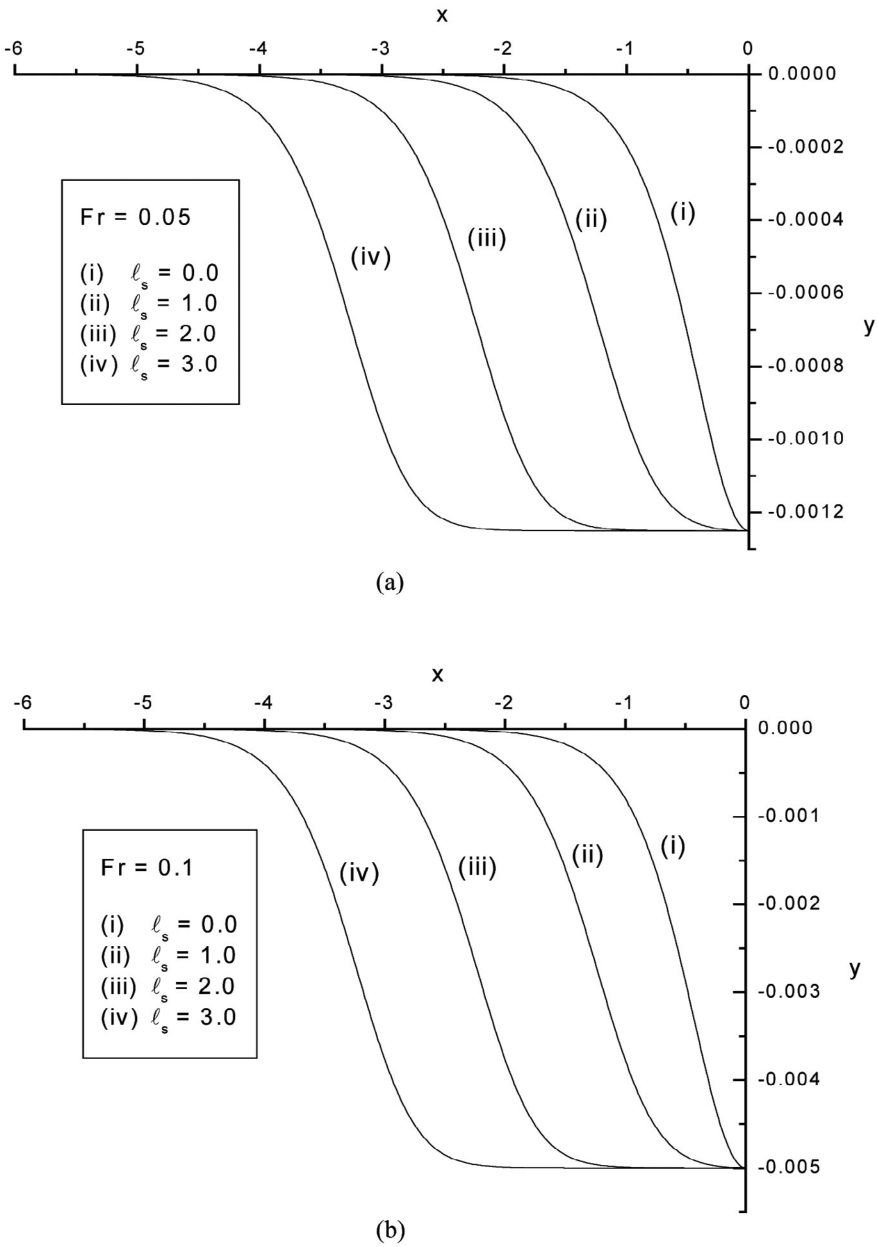


Figure 5.
The free surface flow in a two layer fluid induced by a submerged sink on a solid boundary at $y = 1.0$ when the sink distances are ℓ_s=0, 1.0, 2.0 and 3.0 for (a) Fr = 0.05, and (b) Fr = 0.1

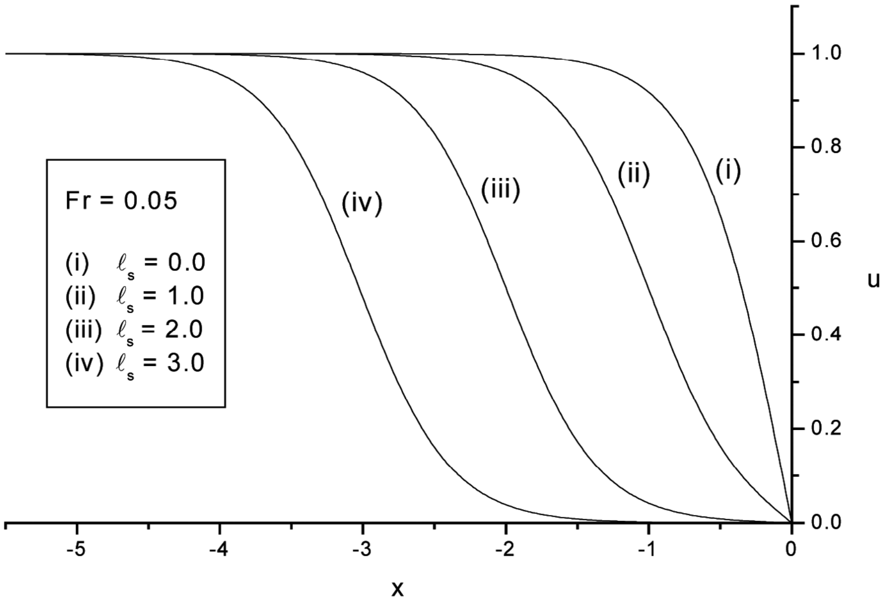


Figure 6.
The free surface velocity for the free surface shown in Figure 5(a) which are induced by a submerged sink on a solid boundary at $y = 1.0$ when the sink distances are $\ell_s = 0$, 1.0, 2.0 and 3.0, for $\mathrm{Fr} = 0.05$, in a two layer fluid

mixing between the two fluids occurs and therefore the critical Froude number decreases.

Figure 6 shows the free surface velocity u for the free surfaces shown in Figure 5(a) which is induced by a submerged sink on a solid boundary at $y = 1.0$ when $\ell_s = 0$, 1.0, 2.0 and 3.0, for $\mathrm{Fr} = 0.05$, in a two layer fluid. We observe that there is a difference in the free surface velocity profiles because the values of ℓ_s are different, similar to the difference in the free surface profiles observed in Figure 5(a). We also note, as ℓ_s increases, the region in which the free surface velocity is very close to zero, or becomes stagnant, also increases. For example, when $\ell_s = 2.0$, the free surface velocity is approximately zero in the region $-0.25 < x \leq 0$ but when $\ell_s = 3.0$ the region in which the free surface is approximately zero has increased to $-1.25 < x \leq 0$.

Critical conditions

We find that on increasing the sink distance ℓ_s, the critical Froude number, for a given value of ℓ_s, decreases. The critical values of the Froude number were found to be approximately 0.29, 0.25, 0.20 and 0.18 when $\ell_s = 0$, 1.0, 2.0 and 3.0, respectively. However, we also note that the value of ℓ_s is dependent on the value of the velocity potential at the point D, i.e. φ_D (Figure 1). Figure 7 shows the relationship between ℓ_s and the velocity potential value φ_D and it is observed that when the value of φ_D is large, e.g. $\varphi_D \approx 5.5$, then $\ell_s \approx 0$. However, to increase the value of ℓ_s it is necessary to reduce the value of φ_D, for example when $\varphi_D = 10^{-8}$, $\ell_s \gg 5.8$. Therefore to find results for $\ell_s \gg 5.8$, it is

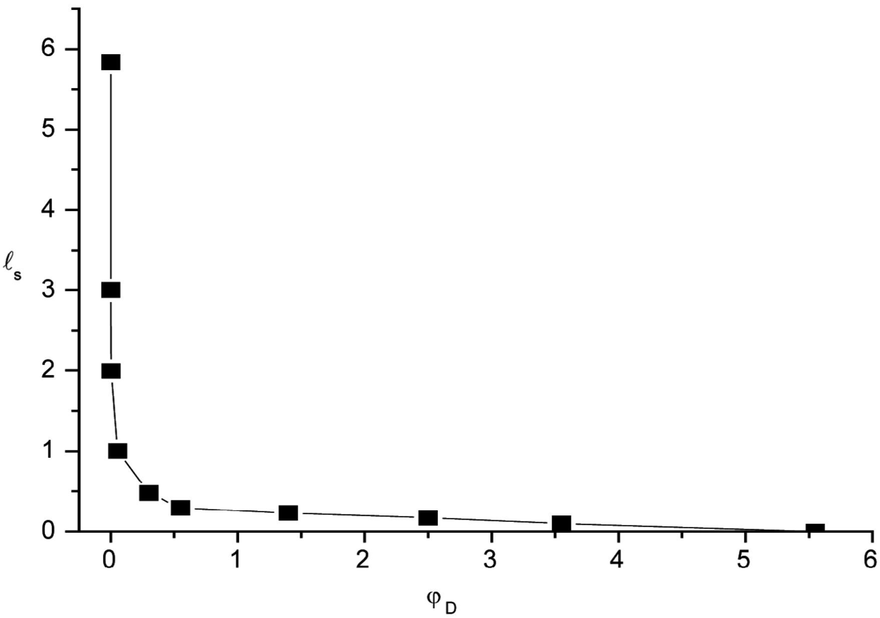


Figure 7.
The relationship between the sink distance ℓ_s and the velocity potential value φ_D

necessary to have $\varphi_D \leqslant 10^{-8}$. Computationally, finding convergent solutions for large values of ℓ_s becomes more difficult for a number of reasons. First, for large values of ℓ_s, say $\ell_s \geqslant 8$, we find that the value of φ_D is very small, typically $\varphi_D \leqslant 10^{-12}$, and convergent solutions become increasingly more difficult to obtain and ultimately when φ_D becomes too small, the numerical scheme breaks down. Secondly, in order to find solutions for large values of ℓ_s, the length $C_\infty D$ also has to increase in order to obtain the full geometry of the free surface profile, which in turn requires a larger number of grid points, N, and therefore increasing the computational time required to obtain convergent solutions.

Conclusions

In this paper, we have developed an accurate and robust boundary integral technique for two-dimensional, inviscid, steady, irrotational free surface fluid flows which are induced by two submerged sinks or sources of equal strength which are situated along a solid horizontal boundary with a stagnation point on the free surface of a two layer fluid in the presence of gravity. The non-linear integral equations derived through the application of the Dirichlet problem, which is a special case of the Riemann-Hilbert problem, have been solved using a boundary integral method which involves the discretisation of the non-linear integral equations and the use of an interpolative technique and an iterative process. The boundary integral technique developed in this paper is different from the traditional boundary element method in that it does not require the inversion of a large system of non-linear algebraic equations and therefore requires less computational memory and CPU time.

This boundary integral technique has been used to study the free surface flow for various Froude numbers and sink locations, ℓ_s, in a two layer fluid. We have found that increasing the sink distance, ℓ_s, has the effect of reducing the value of the critical Froude number, which is the maximum value of the Froude number for which convergent solutions are possible and also this increases the number of iterations required for convergence. As the Froude number increases, we note that the number of iterations increases in order to obtain convergent solutions until the critical Froude number is reached and for any further increase in the Froude number the solution breaks down without any signs of oscillations, i.e. the existence of possible waves occurring on the free surface.

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